Solution to Problem Set 1
Optical Waveguides and Fibers (OWF)

Exercise 1: Long-haul transmission fiber link

For a transatlantic fiber link, e.g., from London to New York, a distance of 7000 km has to be bridged. This is accomplished using optical fibers. Each fiber carries multiple wavelength channels within a total bandwidth that is given by the so-called C-band, which ranges from 1530 nm to 1560 nm. A single wavelength channel transmits data at a data rate of 100 Gbit/s. The spectral separation between two adjacent channels is 50 GHz. The optical fiber used for this link exhibits a loss of 0.2 dB/km within the C-band. These losses are overcome by amplifiers with a gain of $G = 20\,\text{dB}$, where the gain in dB is defined by

$$G = 10 \log \left\{ \frac{P_\text{out}}{P_\text{in}} \right\}.$$

In this equation P_in and P_out are the input and output power of the amplifier, respectively. One amplifier is used per fiber, and all channels are amplified simultaneously. Each amplifier needs electrical power supply. The power efficiency of the amplifier, i.e., the ratio between the total optical output power and the electrical input power amounts to $\eta = 0.01$.

a) How many channels can be used in one fiber? What is the total data rate of the link assuming that it contains 25 fibers?

Hint: Conversion between absolute frequency and wavelength: $f = \frac{c}{\lambda}$. The bandwidth Δf in terms of frequency can be related to bandwidth $\Delta\lambda$ in term of wavelength using: $\Delta f = \frac{df}{d\lambda} \cdot \Delta\lambda$.

Solution:

Conversion between bandwidth in frequency and in wavelength (absolute values): $\Delta f = \frac{c}{\lambda^2} \cdot \Delta\lambda$ and $\Delta\lambda = \frac{c}{f^2} \cdot \Delta f$

$$\Delta f_\text{channel} = 50\,\text{GHz} \implies \Delta\lambda_\text{channel} = 0.4\,\text{nm}$$

$$\Delta\lambda_\text{C-Band} = 30\,\text{nm} \implies \Delta f_\text{C-Band} = 3.7\,\text{THz}$$

A total of 75 channels fit into the C-Band. The total bitrate is:

$$B_\text{tot} = 100\,\frac{\text{Gbit/s}}{\text{channel}} \cdot 75\,\frac{\text{channels}}{\text{fiber}} \cdot 25\,\text{fiber} = 187.5\,\text{Tbit/s}.$$

b) Estimate the total attenuation of an optical signal assuming that there is no amplification along the transmission link. Give the result in logarithmic as well as linear units.

Hint: $a_{[\text{dB}]} = 10 \cdot \log\left(a_{[\text{lin}]}\right)$.

Solution:

$$a_{[\text{dB}]} = 0.2\,\frac{\text{dB}}{\text{km}} \cdot 7000\,\text{km} = 1400\,\text{dB}$$

$$a_{[\text{lin}]} = 10^{140}$$

c) How many photons would you have to launch into the optical fiber in London in order to receive a single photon in New York if not using an optical amplifier? How much energy would be needed to generate this many photons? Compare this energy to the available energy of our universe.

Hint: The (observable) universe is estimated to be a sphere with a radius of 1.4×10^{10} light years, and a mean density of $3 \times 10^{-30}\,\text{g}/\text{cm}^3$.

Solution: For receiving one photon in New York without intermediate amplification, one would have to launch 10^{140} photons into the fiber in London. The energy of one photon can be calculated as $W_\text{ph} = hf = 1.32 \times 10^{-19}\,\text{J}$, considering a frequency of approximately 200 THz. Therefore an energy of $W = 10^{140} \cdot 1.32 \times 10^{-19} = 1.32 \times 10^{121}\,\text{J}$ would be required. The total available energy of the universe could be estimated as $W_\text{univ} = m_\text{univ} c^2 = 3 \times 10^{-24}\,\text{g}/\text{m}^3 \cdot 4\pi/3 \left(1.4 \times 10^{10} \cdot 9.46 \times 10^{15}\,\text{m}\right)^3 \cdot c^2 = 2.7 \times 10^{69}\,\text{J}$. This would mean that one would have to evaporate $\frac{W}{W_\text{univ}} = 5 \times 10^{51}$ universes for sending a single photon without amplification from London to New York.

d) To keep the power distribution along the link as uniform as possible, which distance would you suggest between two amplifiers? How many amplifiers are needed for the link?

Solution: The amplification is 20 dB (a factor 100). This gain is optimally used if the attenuation between two amplifiers is equally high, i.e. after $^{20\,\mathrm{dB}}/_{0.2\,\mathrm{dB/km}} = 100\,\mathrm{km}$. For the whole distance, a total of 70 amplifiers would be required.

e) Estimate the power consumption assuming that each wavelength channel should carry a power of 0 dBm at the output of each amplifier. Assume that electrical line losses can be neglected.

Hint: $P_{[\mathrm{dBm}]} = 10 \log \left\{ \frac{P_{[\mathrm{lin}]}}{1\mathrm{mW}} \right\}$

Solution: Power per channel: 0 dBm$\hat{=}$1mW

Power per fiber and amplifier: 75 mW

Power efficiency: $\eta = \frac{P_{\mathrm{out.opt}}}{P_{\mathrm{in,el}}} = 0.01$

Total power consumption: $P_{\mathrm{in,el,tot}} = 70\,\mathrm{amplifiers} \cdot 75\,^{\mathrm{mW}}/_{\mathrm{amplifier/fiber}} \cdot 25\,\mathrm{fibers}/0.01 = 13.1\,\mathrm{kW}$

Exercise 2: Helmholtz equation for the magnetic field $\underline{\mathbf{H}}$

Assuming a time dependence of the kind $e^{j\omega t}$, and assuming that there are no free charges, no free currents and non-magnetic medium, Maxwell's equations take the following form:

$$\begin{aligned} \nabla \cdot \underline{\mathbf{D}} &= 0 & (1) \\ \nabla \times \underline{\mathbf{H}} &= j\omega\epsilon_0 \underline{\epsilon}_r \underline{\mathbf{E}} & (2) \\ \nabla \times \underline{\mathbf{E}} &= -j\omega\mu_o \underline{\mathbf{H}} & (3) \\ \nabla \cdot \underline{\mathbf{H}} &= 0. & (4) \end{aligned}$$

Vectors are denoted in **bold** and complex quantities are underlined.

a) Derive the vector wave equation for the magnetic field in inhomogeneous media:

$$\nabla^2 \underline{\mathbf{H}} + \frac{\nabla \underline{\epsilon}_r}{\underline{\epsilon}_r} \times (\nabla \times \underline{\mathbf{H}}) + \omega^2 \mu_0 \epsilon_0 \underline{\epsilon}_r \underline{\mathbf{H}} = 0 \qquad (5)$$

Hints:

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} & (6) \\ \nabla \times (\psi \mathbf{A}) &= (\nabla \psi) \times \mathbf{A} + \psi \nabla \times \mathbf{A} & (7) \end{aligned}$$

Solution: Taking the rotor of Eq. (2) and using both hints we get:

$$\nabla (\nabla \underline{\mathbf{H}}) - \nabla^2 \underline{\mathbf{H}} = j\omega\epsilon_0 \left[(\nabla \underline{\epsilon}_r) \times \underline{\mathbf{E}} + \underline{\epsilon}_r \nabla \times \underline{\mathbf{E}} \right]$$

The first term is zero because of Eq. (4). Substituting now the electric field and its rotor thanks to Eqs. (2) and (3) respectively, we get:

$$-\nabla^2 \underline{\mathbf{H}} = j\omega\epsilon_0 \left[\nabla(\underline{\epsilon}_r) \times \left(\frac{\nabla \times \underline{\mathbf{H}}}{j\omega\epsilon_0 \underline{\epsilon}_r} \right) + \underline{\epsilon}_r \left(-j\omega\mu_0 \underline{\mathbf{H}} \right) \right].$$

Reordering the terms leads to Eq. (5).

b) Show that if $\nabla \underline{\epsilon}_r = 0$, then $\underline{\mathbf{H}} = \underline{\mathbf{H}}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}$ is a solution of Eq. (5) with $\mathbf{k}^2 = \omega^2 \epsilon_0 \underline{\epsilon}_r \mu_0$.

Solution: Substituting the Ansatz into Eq. (5) leads to:

$$\begin{aligned} -\mathbf{k}^2 \underline{\mathbf{H}} + \omega^2 \mu_0 \epsilon_0 \underline{\epsilon}_r \underline{\mathbf{H}} &= 0 \\ -\mathbf{k}^2 + \omega^2 \mu_0 \epsilon_0 \underline{\epsilon}_r &= 0 \quad . \end{aligned}$$

c) Show that $\mathbf{k}$ and $\underline{\mathbf{H}}_0$ must be orthogonal.

Solution: Applying Eq. (4) to the Ansatz we get $\mathbf{k} \cdot \underline{\mathbf{H}} = 0$, which shows that $\mathbf{k}$ and $\underline{\mathbf{H}}_0$ are orthogonal.

d) Assume that $\underline{\mathbf{H}}_0 = H_0\mathbf{e}_y$ and $\mathbf{k} = k\mathbf{e}_z$, where $\mathbf{e}_y$ ($\mathbf{e}_z$) is the unit vector along y (z). Calculate the corresponding electric field. Observe that the vectors $\underline{\mathbf{E}}$, $\underline{\mathbf{H}}$ and $\mathbf{k}$ of the previous point are mutually orthogonal and form a right-handed triple.

Solution: The electric field can be calculated using Eq. (2). Inserting the Ansatz leads to:

$$\underline{\mathbf{E}} = \frac{1}{j\omega\epsilon_0\underline{\epsilon}_r}\nabla \times \underline{\mathbf{H}} = -\frac{kH_0}{\omega\epsilon_0\underline{\epsilon}_r}e^{-jkz}\mathbf{e}_z \times \mathbf{e}_y.$$

The vectors $\underline{\mathbf{E}}$, $\underline{\mathbf{H}}$ and $\mathbf{k}$ form a right-handed triple since $\mathbf{e}_z \times \mathbf{e}_y = -\mathbf{e}_x$.

e) If using the Ansatz $\underline{\mathbf{H}} = \underline{\mathbf{H}}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}$ in Eq. (5), which condition must be satisfied by the normalized spatial variation of the dielectric constant $\nabla\underline{\epsilon}_r/\underline{\epsilon}_r$ such that the second term in Eq. (5) is negligible compared to the third term?

Solution: In order to neglect the second term in Eq. (5) we have to show that:

$$\left|\frac{\nabla\underline{\epsilon}_r}{\underline{\epsilon}_r} \times (\nabla \times \underline{\mathbf{H}})\right| \ll \left|\omega^2\mu_0\epsilon_0\underline{\epsilon}_r\underline{\mathbf{H}}\right|.$$

Substituting the Ansatz we get:

$$\left|\frac{\nabla\underline{\epsilon}_r}{\underline{\epsilon}_r} \times (-j\mathbf{k} \times \underline{\mathbf{H}}_0)\right| \ll \left|\omega^2\mu_0\epsilon_0\underline{\epsilon}_r\underline{\mathbf{H}}_0\right|.$$

Sufficient condition for the latter equation is that:

$$\left|\frac{\nabla\underline{\epsilon}_r}{\underline{\epsilon}_r}\right| \cdot |\mathbf{k}| \cdot |\underline{\mathbf{H}}_0| \ll \left|\mathbf{k}^2\underline{\mathbf{H}}_0\right|,$$

where $\mathbf{k}^2 = \omega^2\epsilon_0\underline{\epsilon}_r\mu_0$ has been used. This leads to the condition

$$\left|\frac{\nabla\underline{\epsilon}_r}{\underline{\epsilon}_r}\right| \ll |\mathbf{k}| = \frac{2\pi}{\lambda},$$

which means that the relative dielectric constant must vary little over the wavelength.

Questions and Comments:

Aleksandar Nesic
Building: 30.10, Room: 2.32-2
Phone: 0721/608-42480
aleksandar.nesic@kit.edu

Philipp Trocha
Building: 30.10, Room: 2.32-2
Phone: 0721/608-42480
philipp.trocha@kit.edu